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Markov- and Bernstein-Type Inequalities for Müntz Polynomials and Exponential Sums in L_p

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The principal result of this paper is the following Markov-type inequality for Müntz polynomials.

THEOREM (Newman's Inequality in $L_p[a, b]$ for $[a, b] \subset (0, \infty)$). *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be an increasing sequence of nonnegative real numbers. Suppose $\lambda_0 = 0$ and there exists a $\delta > 0$ so that $\lambda_j \geqslant \delta j$ for each j. Suppose $0 < a < b$ and $1 \leqslant p \leqslant \infty$. Then there exists a constant $c(a, b, \delta)$ depending only on a, b, and δ so that $\|P'\|_{L_p[a, b]} \leqslant c(a, b, \delta)(\sum_{j=0}^{n} \lambda_j) \|P\|_{L_p[a, b]}$ for every $P \in M_n(\Lambda)$, where $M_n(\Lambda)$ denotes the linear span of $\{x^{\lambda_0}, x^{\lambda_1}, ..., x^{\lambda_n}\}$ over $\mathbb{R}$.*

When $p = \infty$ this has been shown by P. B. Borwein and the author (1996, *J. Approx. Theory* **85**, 132–139). When $[a, b] = [0, 1]$ and with $\|P'\|_{L_p[a, b]}$ replaced with $\|xP'(x)\|_{L_p[a, b]}$ this was proved by D. Newman (1976, *J. Approx. Theory* **18**, 360–362) for $p = \infty$ and by P. Borwein and the author (1996, *Proc. Amer. Math. Soc.* **124**, 101–109) for $1 \leqslant p \leqslant \infty$. Note that the interval $[0, 1]$ plays a special role in the study of Müntz spaces $M_n(\Lambda)$. A linear transformation $y = \alpha x + \beta$ does not preserve membership in $M_n(\Lambda)$ in general (unless $\beta = 0$). So the analogue of Newman's Inequality on $[a, b]$ for $a > 0$ does not seem to be obtainable in any straightforward fashion from the $[0, b]$ case.

Key Words: Müntz polynomials; lacunary polynomials; exponential sums; Dirichlet sums; Markov-type inequality; Bernstein-type inequality.

1. INTRODUCTION AND NOTATION

Let $\mathscr{P}_n$ denote the collection of all algebraic polynomials of degree at most n with real coefficients. For notational convenience let $\|\cdot\|_{[a, b]} := \|\cdot\|_{L_\infty[a, b]}$. The following two inequalities, together with their various extensions, play an important role in approximation theory. See, for example, DeVore and Lorentz [8], Lorentz [10], and Natanson [12].

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THEOREM 1.1 (Markov's Inequality). *If $p \in \mathcal{P}_n$, then*

$$\|p'\|_{[-1,1]} \leqslant n^2 \|p\|_{[-1,1]}.$$

THEOREM 1.2 (Bernstein's Inequality). *If $p \in \mathcal{P}_n$, then*

$$|p'(x)| \leqslant \frac{n}{\sqrt{1-x^2}} \|p\|_{[-1,1]}, \qquad -1 < x < 1.$$

Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct real numbers. The linear span of

$$\{x^{\lambda_0}, x^{\lambda_1}, ..., x^{\lambda_n}\}$$

over $\mathbb{R}$ will be denoted by

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ..., x^{\lambda_n}\}.$$

Elements of $M_n(\Lambda)$ are called Müntz polynomials.

Newman's inequality [13] is an essentially sharp Markov-type inequality for $M_n(\Lambda)$, where $\Lambda := (\lambda_j)_{j=0}^{\infty}$ is a sequence of distinct nonnegative real numbers.

THEOREM 1.3 (Newman's Inequality). *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct nonnegative real numbers. Then*

$$\frac{2}{3} \sum_{j=0}^{n} \lambda_j \leqslant \sup_{0 \neq P \in M_n(\Lambda)} \frac{\|xP'(x)\|_{[0,1]}}{\|P\|_{[0,1]}} \leqslant 11 \sum_{j=0}^{n} \lambda_j.$$

Frappier [9] shows that the constant 11 in Newman's inequality can be replaced by 8.29. In [3], by modifying (and simplifying) Newman's arguments, we showed that the constant 11 in the above inequality can be replaced by 9. But more importantly, this modification allowed us to prove the following L_p version of Newman's inequality [3] (an L_2 version of which was proved earlier in [6]).

THEOREM 1.4 (Newman's Inequality in $L_p[0,1]$). *Let $1 \leqslant p \leqslant \infty$. Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct real numbers greater than $-1/p$. Then*

$$\|xP'(x)\|_{L_p[0,1]} \leqslant \left(1/p + 12\left(\sum_{j=0}^{n} (\lambda_j + 1/p)\right)\right) \|P\|_{L_p[0,1]}$$

for every $P \in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ..., x^{\lambda_n}\}$.

In this paper, using the fact that the constant 11 in Theorem 1.3 can be replaced by 8.29, we will show that the constant 12 in Theorem 1.4 can be replaced by 8.29 as well. See Theorems 2.1 and 2.2.

On the basis of considerable computation, in [3] we speculate that the best possible constant in Newman's inequality is 4. (We remark that an incorrect argument exists in the literature claiming that the best possible constant in Newman's inequality is at least $4+\sqrt{15}=7.87...$)

It is proved in [2] that under a growth condition, which is essential, $\|xP'(x)\|_{[0,1]}$ in Newman's inequality can be replaced by $\|P'\|_{[0,1]}$. More precisely, the following result holds.

THEOREM 1.5 (Newman's Inequality without the Factor x). *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct real numbers with $\lambda_0=0$ and $\lambda_j \geqslant j$ for each j. Then*

$$\|P'\|_{[0,1]} \leqslant 16.58 \left(\sum_{j=1}^{n} \lambda_j \right) \|P\|_{[0,1]}$$

for every $P \in M_n(\Lambda)$.

Note that the interval $[0, 1]$ plays a special role in the study of Müntz polynomials. A linear transformation $y=\alpha x+\beta$ does not preserve membership in $M_n(\Lambda)$ in general (unless $\beta=0$), that is $P \in M_n(\Lambda)$ does not necessarily imply that $Q(x) := P(\alpha x+\beta) \in M_n(\Lambda)$. Analogues of the above results on $[a, b]$, $a>0$, cannot be obtained by a simple transformation. Nevertheless in [5], under a growth condition, which is essential, we have established a version of Newman's inequality on intervals $[a, b]$, $a>0$. Here we prove an analogue of this result in $L_p[a, b]$ with $a>0$ and $1 \leqslant p < \infty$.

The rational functions and exponential sums belong to those concrete families of functions which are the most frequently used in nonlinear approximation theory. See, for example, Braess [7]. The starting point of consideration of exponential sums is an approximation problem often encountered for the analysis of decay processes in natural sciences. A given empirical function on a real interval is to be approximated by sums of the form

$$\sum_{j=1}^{n} a_j e^{\lambda_j t},$$

where the parameters a_j and λ_j are to be determined, while n is fixed.

In [4] we proved the "right" Bernstein-type inequality for exponential sums. This inequality is the key to proving inverse theorems for approximation by exponential sums. Let

$$E_n := \left\{ f \colon f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t},\ a_j, \lambda_j \in \mathbb{R} \right\}.$$

So E_n is the collection of all $n+1$ term exponential sums with constant first term. Schmidt [14] proved that there is a constant $c(n)$ depending only on n so that

$$\|f'\|_{[a+\delta,\, b-\delta]} \leqslant c(n)\, \delta^{-1}\, \|f\|_{[a,b]}$$

for every $f \in E_n$ and $\delta \in (0, \frac{1}{2}(b-a))$. Lorentz [11] improved Schmidt's result by showing that for every $\alpha > \frac{1}{2}$, there is a constant $c(\alpha)$ depending only on α so that $c(n)$ in the above inequality can be replaced by $c(\alpha)\, n^{\alpha \log n}$ (Xu improved this to allow $\alpha = \frac{1}{2}$), and he speculated that there may be an absolute constant c so that Schmidt's inequality holds with $c(n)$ replaced by cn. We [1] proved a weaker version of this conjecture with cn^3 instead of cn. The main result of [4] shows that Schmidt's inequality holds with $c(n) = 2n-1$. This essentially sharp result can also be formulated as

THEOREM 1.6. *We have*

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a,\, b-y\}} \leqslant \sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leqslant \frac{2n-1}{\min\{y-a,\, b-y\}}$$

for all $y \in (a, b)$.

This result complements Newman's Markov-type inequality (see [13] and [5]) given by Theorem 1.3. In this paper we establish an L_p version of Theorem 1.6. See Theorem 3.4.

Bernstein-type inequalities play a very important role in approximation theory via a machinery developed by Bernstein, which turns Bernstein-type inequalities into inverse theorems of approximation. See, for example, Lorentz [10] and DeVore and Lorentz [8].

2. NEW RESULTS: NEWMAN'S INEQUALITY IN $L_p[0, 1]$ WITH THE CONSTANT 8.29

THEOREM 2.1. *Let* $1 \leqslant p \leqslant \infty$. *Let* $\Lambda := (\lambda_j)_{j=0}^{\infty}$ *be a sequence of distinct real numbers greater than* $-1/p$. *Then*

$$\|xS'(x)\|_{L_p[0,1]} \leqslant \left(1/p + 8.29 \left(\sum_{j=0}^{n} (\lambda_j + 1/p)\right)\right) \|S\|_{L_p[0,1]}$$

for every $S \in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$.

THEOREM 2.2. *Let $1 \leqslant p \leqslant \infty$. Let $\Gamma := (\gamma_j)_{j=0}^{\infty}$ be a sequence of distinct positive numbers. Then*

$$\|Q'\|_{L_p[0,\infty)} \leqslant 8.29 \left(\sum_{j=0}^{n} \gamma_j \right) \|Q\|_{L_p[0,\infty)}$$

for every $Q \in E_n(\Gamma) := \operatorname{span}\{e^{-\gamma_0 t}, e^{-\gamma_1 t}, ..., e^{-\gamma_n t}\}$.

The $L_\infty[0, 1]$ version of the above inequalities are due to Newman [13] with the constant 11 rather than 8.29. The $L_\infty[0, 1]$ version of the above inequalities is proved in [9]. A slightly simplified version of Newman's proof in the $L_\infty[0, 1]$ case as well as the above $L_p[0, 1]$ inequalities with the constant 12 rather than 8.29 are given in both [2, 3]. Here we will reduce the proof of the above $L_p[0, 1]$ inequalities to Newman's inequality given by Theorem 1.3, by recalling, as we have already remarked, that the constant 11 in Theorem 1.3 can be replaced by 8.29.

3. NEW RESULTS: NEWMAN'S INEQUALITY IN $L_p[a, b]$ FOR $[a, b] \subset (0, \infty)$

We establish two Markov-type inequalities, one for $M_n(\Lambda)$ in $L_p[a, b]$ for $[a, b] \subset (0, \infty)$, and one for $E_n(\Gamma)$ in $L_p[a, b]$ for $[a, b] \subset (-\infty, \infty)$. It is very simple to see that these follow from each other.

THEOREM 3.1 (Markov Inequality for $M_n(\Lambda)$ in $L_p[a, b]$). *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be an increasing sequence of nonnegative real numbers. Suppose $\lambda_0 = 0$ and there exists a $\delta > 0$ so that $\lambda_j \geqslant \delta j$ for each j. Suppose $0 < a < b < \infty$ and $1 \leqslant p \leqslant \infty$. Then there exists a constant $c(a, b, \delta)$ depending only on a, b, and δ so that*

$$\|P'\|_{L_p[a,b]} \leqslant c(a, b, \delta) \left\{ \sum_{j=0}^{n} \lambda_j \right\} \|P\|_{L_p[a,b]}$$

for every $P \in M_n(\Lambda)$, where $M_n(\Lambda)$ denotes the linear span of $\{x^{\lambda_0}, x^{\lambda_1}, ..., x^{\lambda_n}\}$ over $\mathbb{R}$.

THEOREM 3.2 (Markov Inequality for $E_n(\Lambda)$ in $L_p[a, b]$). *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be an increasing sequence of nonnegative real numbers. Suppose $\lambda_0 = 0$ and there exists a $\delta > 0$ so that $\lambda_j \geqslant \delta j$ for each j. Suppose*

$-\infty < a < b < \infty$ *and* $1 \leqslant p \leqslant \infty$. *Then there exists a constant* $c(a, b, \delta)$ *depending only on* a, b, *and* δ *so that*

$$\|P'\|_{L_p[a, b]} \leqslant c(a, b, \delta) \left(\sum_{j=0}^{n} \lambda_j \right) \|P\|_{L_p[a, b]}$$

for every $P \in E_n(\Lambda)$, *where* $E_n(\Lambda)$ *denotes the linear span of* $\{e^{\lambda_0 t}, e^{\lambda_1 t}, ..., e^{\lambda_n t}\}$ *over* $\mathbb{R}$.

The $p = \infty$ case of Theorem 3.1 is proved in [5]. The proof of the general case will be reduced to this one. Notice that Theorem 3.1 follows from Theorem 3.2 by the substitution $x = e^t$. Therefore we need to prove only Theorem 3.2. Observe also that the $p = \infty$ case of Theorem 3.2 follows from the $p = \infty$ case of Theorem 3.1, so it is sufficient to reduce the general case to this one again.

The following example shows that the growth condition $\lambda_j \geqslant \delta j$ with a $\delta > 0$ in the above theorem cannot be dropped. It has been used in [5] as well.

THEOREM 3.3. *Let* $\Lambda := (\lambda_j)_{j=0}^{\infty}$, *where* $\lambda_j = \delta j$. *Let* $0 < a < b$. *Then*

$$\max_{0 \neq P \in M_n(\Lambda)} \frac{|P'(a)|}{\|P\|_{[a, b]}} = |Q_n'(a)| = \frac{2\delta a^{\delta - 1}}{b^\delta - a^\delta} n^2,$$

where, with $T_n(x) = \cos(n \arccos x)$,

$$Q_n(x) := T_n \left(\frac{2x^\delta}{b^\delta - a^\delta} - \frac{b^\delta + a^\delta}{b^\delta - a^\delta} \right)$$

is the Chebyshev "polynomial" for $M_n(\Lambda)$ *on* $[a, b]$. *In particular*

$$\lim_{\delta \to 0} \max_{0 \neq P \in M_n(\Lambda)} \frac{|P'(a)|}{(\sum_{j=0}^{n} \lambda_j) \|P\|_{[a, b]}} = \infty.$$

Theorem 3.3 is a well-known property of differentiable Chebyshev spaces. See, for example, [2, 5].

Finally we record the extension of Theorem 1.6 to $L_p[a, b]$ spaces. Note that no assumptions on the set of exponents are prescribed.

THEOREM 3.4 (Bernstein Inequality in $L_p[a, b]$ for E_n). *Let* $\delta \in (0, (b-a)/2)$. *We have*

$$\sup_{0 \neq f \in E_n} \frac{\|f'\|_{L_p[a+\delta, b-\delta]}}{\|f\|_{L_p[a, b]}} \leqslant \frac{2n-1}{\delta}.$$

4. AN INTERPOLATION THEOREM

To reduce the $1 \leqslant p \leqslant \infty$ case of Theorems 2.2, 3.2, and 3.4 to the $p = \infty$ case, the main tool is the Interpolation Theorem below. See [2, p. 385].

Interpolation of Linear Functionals. Let $C(Q)$ be the set of real- (complex-) valued continuous functions on the compact Hausdorff space Q. Let S be an n-dimensional linear subspace of $C(Q)$ over $\mathbb{R}$ ($\mathbb{C}$). Let $L \neq 0$ be a real- (complex-) valued linear functional on S. Then there exist points $x_1, x_2, \ldots, x_r$ in Q and nonzero real (complex) numbers $a_1, a_2, \ldots, a_r$, where $1 \leqslant r \leqslant n$ in the real case and $1 \leqslant r \leqslant 2n-1$ in the complex case, such that

$$L(s) = \sum_{i=1}^{r} a_i s(x_i), \qquad s \in S,$$

and

$$\|L\| = \sup\{|L(s)|: s \in S, \|s\|_Q \leqslant 1\} = \sum_{i=1}^{r} |a_i|.$$

5. PROOFS

First we show that Theorem 2.1 follows from Theorem 2.2. Indeed, assume that $\lambda_0, \lambda_1, \ldots, \lambda_n$ are distinct real numbers greater than $-1/p$. Let

$$S \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}.$$

Then $\gamma_i := \lambda_i + 1/p$ $(i = 0, 1, \ldots, n)$ are distinct positive numbers. Applying Theorem 2.2 with

$$Q(t) := S(e^{-t})\, e^{-t/p} \in \operatorname{span}\{e^{-\gamma_0 t}, e^{-\gamma_1 t}, \ldots, e^{-\gamma_n t}\}$$

and using the substitution $x = e^{-t}$, we obtain

$$\int_0^1 |x(x^{1/p} S(x))'|^p\, x^{-1}\, dx \leqslant 8.29 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right)^p \int_0^1 |S(x)|^p\, dx.$$

Now the product rule of differentiation and Minkowski's inequality yield

$$\int_0^1 |xS'(x)|^p\, dx \leqslant \left(1/p + 8.29 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right) \right)^p \int_0^1 |S(x)|^p\, dx,$$

which is the inequality of Theorem 2.1. Now we prove Theorem 2.2.

Proof of Theorem 2.2. Note that the fact that $E_n(\Gamma)$ is a finite dimensional vector space implies that there is a $b>0$ such that

$$\|s\|_{[0,\infty)} \leqslant \|s\|_{[0,b]}$$

for every $s \in E_n(\Gamma)$. We apply the Interpolation Theorem of Section 4 with $Q := [0, b]$, $S := E_n(\Gamma)$, and $L(s) := s'(0)$. As we have already remarked, Theorem 1.3 (Newman's inequality) holds with the constant 8.29 rather than 11. This implies that

$$\|L\| \leqslant c(\Gamma) := 8.29 \left(\sum_{j=0}^{n} \gamma_j \right).$$

We deduce that there are $x_1, x_2, ..., x_r$ in $[0, b]$ and $c_1, c_2, ..., c_r \in \mathbb{R}$ so that for every $s \in E_n(\Gamma)$ we have

$$\frac{|s'(0)|}{c(\Gamma)} \leqslant \left| \sum_{i=1}^{r} c_i s(x_i) \right| \leqslant \sum_{i=1}^{r} |c_i| \, |s(x_i)|$$

with $\sum_{i=1}^{r} |c_i| = 1$ and $1 \leqslant r \leqslant n+1$. Now let $\varphi\colon [0, \infty) \mapsto [0, \infty)$ be a nondecreasing convex function. Using monotonicity and convexity, we obtain

$$\varphi\left(\frac{|s'(0)|}{c(\Gamma)}\right) \leqslant \varphi\left(\sum_{i=1}^{r} |c_i| \, |s(x_i)|\right) \leqslant \sum_{i=1}^{r} |c_i| \, \varphi(|s(x_i)|).$$

Applying this with $s(t) := P(t+y) \in E_n(\Gamma)$, we deduce

$$\varphi\left(\frac{|P'(y)|}{c(\Gamma)}\right) \leqslant \sum_{i=1}^{r} |c_i| \, \varphi(|P(x_i+y)|)$$

for every $P \in E_n(\Gamma)$ and $y \in [0, \infty)$, where $x_i \in [0, b]$ and $y \in [0, \infty)$ imply that $x_i + y \in [0, \infty)$ for each $i = 1, 2, ..., r$. Integrating on the interval $[0, \infty)$ with respect to y, we obtain

$$\int_0^\infty \varphi\left(\frac{|P'(y)|}{c(\Gamma)}\right) dy \leqslant \sum_{i=1}^{r} \int_0^\infty |c_i| \, \varphi(|P(x_i+y)|) \, dy$$

$$\leqslant \sum_{i=1}^{r} \int_0^\infty |c_i| \, \varphi(|P(t)|) \, dt \leqslant \int_0^\infty \varphi(|P(t)|) \, dt,$$

where $\sum_{i=1}^{r} |c_i| = 1$ has been used. Now the choice of $\varphi(x) := x^p$ $(1 \leqslant p < \infty)$ gives the theorem. ∎

Now we prove Theorem 3.2 (see the remark after Theorem 3.2).

Proof of Theorem 3.2. Let $c := (a+b)/2$. We apply the Interpolation Theorem of Section 4 with $Q := [c, b]$, $S := E_n(\Lambda)$, and $L(s) := s'(b)$. As we have already remarked, the L_∞ case of the theorem has been proved in [5]. This yields that

$$\|L\| \leqslant c(a, b, \delta, \Lambda) := c(a, b, \delta)\left(\sum_{j=0}^{n} \lambda_j\right).$$

We deduce that there are $x_1, x_2, ..., x_r$ in $[c, b]$ and $c_1, c_2, ..., c_r \in \mathbb{R}$ so that for every $s \in E_n(\Lambda)$ we have

$$\frac{|s'(b)|}{c(a, b, \delta, \Lambda)} \leqslant \left|\sum_{i=1}^{r} c_i s(x_i)\right| \leqslant \sum_{i=1}^{r} |c_i|\, |s(x_i)|$$

with $\sum_{i=1}^{r} |c_i| = 1$ and $1 \leqslant r \leqslant n+1$. Now let φ: $[0, \infty) \mapsto [0, \infty)$ be a non-decreasing convex function. Using monotonicity and convexity, we obtain

$$\varphi\left(\frac{|s'(b)|}{c(a, b, \delta, \Lambda)}\right) \leqslant \varphi\left(\sum_{i=1}^{r} |c_i|\, |s(x_i)|\right) \leqslant \sum_{i=1}^{r} |c_i|\, \varphi(|s(x_i)|).$$

Applying this with $s(t) := P(t+y-b) \in E_n(\Lambda)$, we deduce

$$\varphi\left(\frac{|P'(y)|}{c(a, b, \delta, \Lambda)}\right) \leqslant \sum_{i=1}^{r} |c_i|\, \varphi(|P(x_i+y-b)|)$$

for every $P \in E_n(\Lambda)$ and $y \in [c, b]$, where $x_i \in [c, b]$ and $y \in [c, b]$ imply that $x_i + y - b \in [a, b]$ for each $i = 1, 2, ..., r$. Integrating on the interval $[c, b]$ with respect to y, we obtain

$$\begin{aligned}\int_c^b \varphi\left(\frac{|P'(y)|}{c(a, b, \delta, \Lambda)}\right) dy &\leqslant \sum_{i=1}^{r} \int_c^b |c_i|\, \varphi(|P(x_i+y-b)|)\, dy \\ &\leqslant \sum_{i=1}^{r} \int_a^b |c_i|\, \varphi(|P(t)|) \leqslant \int_a^b \varphi(|P(t)|)\, dt,\end{aligned}$$

where $\sum_{i=1}^{r} |c_i| = 1$ has been used. It can be shown exactly in the same way that

$$\int_a^c \varphi\left(\frac{|P'(y)|}{c(a, b, \delta, \Lambda)}\right) dy \leqslant \int_a^b \varphi(|P(t)|)\, dt.$$

Combining the last two inequalities and choosing $\varphi(x) := x^p$ $(1 \leqslant p < \infty)$, we conclude the theorem. ∎

Proof of Theorem 3.4. We apply the Interpolation Theorem of Section 4 with $Q := [-\delta, \delta]$, $S := E_n(\Lambda)$, and $L(s) := s'(0)$. The L_∞ case of the theorem is given by Theorem 1.6. This yields that

$$\|L\| \leqslant \frac{2n-1}{\delta}.$$

We deduce that there are $x_1, x_2, ..., x_r$ in $[-\delta, \delta]$ and $c_1, c_2, ..., c_r \in \mathbb{R}$ so that for every $s \in E_n(\Lambda)$ we have

$$\frac{|s'(0)|\ \delta}{2n-1} \leqslant \left| \sum_{i=1}^{r} c_i s(x_i) \right| \leqslant \sum_{i=1}^{r} |c_i|\ |s(x_i)|$$

with $\sum_{i=1}^r |c_i| = 1$ and $1 \leqslant r \leqslant n+1$. Now let φ: $[0, \infty) \mapsto [0, \infty)$ be a nondecreasing convex function. Using monotonicity and convexity, we obtain

$$\varphi\left(\frac{|s'(0)|\ \delta}{2n-1}\right) \leqslant \varphi\left(\sum_{i=1}^{r} |c_i|\ |s(x_i)|\right) \leqslant \sum_{i=1}^{r} |c_i|\ \varphi(|s(x_i)|).$$

Applying this with $s(t) := P(t+y) \in E_n(\Lambda)$, we deduce

$$\varphi\left(\frac{|P'(y)|\ \delta}{2n-1}\right) \leqslant \sum_{i=1}^{r} |c_i|\ \varphi(|P(x_i+y)|)$$

for every $P \in E_n(\Lambda)$ and $y \in [a+\delta, b-\delta]$, where $x_i \in [-\delta, \delta]$ and $y \in [a+\delta, b-\delta]$ imply that $x_i + y \in [a, b]$ for each $i = 1, 2, ..., r$. Integrating on the interval $[a+\delta, b-\delta]$ with respect to y, we obtain

$$\begin{aligned}\int_{a+\delta}^{b-\delta} \varphi\left(\frac{|P'(y)|\ \delta}{2n-1}\right) dy &\leqslant \sum_{i=1}^{r} \int_{a+\delta}^{b-\delta} |c_i|\ \varphi(|P(x_i+y)|)\, dy \\ &\leqslant \sum_{i=1}^{r} \int_{a}^{b} |c_i|\ \varphi(|P(t)|)\, dt \leqslant \int_{a}^{b} \varphi(|P(t)|)\, dy,\end{aligned}$$

where $\sum_{i=1}^r |c_i| = 1$ has been used. Choosing now $\varphi(x) := x^p$ $(1 \leqslant p < \infty)$, we conclude the theorem. ∎

REFERENCES

1. P. B. Borwein and T. Erdélyi, Upper bounds for the derivative of exponential sums, *Proc. Amer. Math. Soc.* **123** (1995), 1481–1486.
2. P. B. Borwein and T. Erdélyi, "Polynomials and Polynomials Inequalities," Springer-Verlag, New York, 1995.
3. P. B. Borwein and T. Erdélyi, The L_p version of Newman's inequality for lacunary polynomials, *Proc. Amer. Math. Soc.* **124** (1996), 101–109.

4. P. B. Borwein and T. Erdélyi, A sharp Bernstein-type inequality for exponential sums, *J. Reine Angew. Math.* **476** (1996), 127–141.
5. P. B. Borwein and T. Erdélyi, Newman's inequality for Müntz polynomials on positive intervals, *J. Approx. Theory* **85** (1996), 132–139.
6. P. B. Borwein, T. Erdélyi, and J. Zhang, Müntz systems and orthogonal Müntz–Legendre polynomials, *Trans. Amer. Math. Soc.* **342** (1994), 523–542.
7. D. Braess, "Nonlinear Approximation Theory," Springer-Verlag, Berlin, 1986.
8. R. A. DeVore and G. G. Lorentz, "Constructive Approximation," Springer-Verlag, Berlin, 1993.
9. C. Frappier, "Quelques problèmes extrémaux pour les polynomes et les fonctions entières de type exponentiel," Ph.D. Dissertation, Université de Montreal, 1982.
10. G. G. Lorentz, "Approximation of Functions," 2nd ed., Chelsea, New York, 1986.
11. G. G. Lorentz, Notes on approximation, *J. Approx. Theory* **56** (1989), 360–365.
12. I. P. Natanson, "Constructive Function Theory," Vol. 1, Ungar, New York, 1964.
13. D. J. Newman, Derivative bounds for Müntz polynomials, *J. Approx. Theory* **18** (1976), 360–362.
14. E. Schmidt, Zur Kompaktheit der Exponentialsummen, *J. Approx. Theory* **3** (1970), 445–459.